# Lemonade and Cookies – A Beginning Linear Programming Problem

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## Introduction

Linear programming has been called "the most important real world mathematics problem." Essential to every major shipping, distribution, transportation, telecommunication, large retail, and production company, this tool's importance cannot be understated.

So what is linear programming? **Linear programming** is a method of determining how to allocate scarce resources among competing needs to determine an optimum outcome.

The example we explore here is a lemonade and cookie stand where the amount of sugar and the time needed to make the goodies are the scarce resources.

## Importance of Linear Programming

Linear programming was developed during the Second World War by George B. Danzig and others to solve military optimization problems – including rations for troops. Today this tool and its variants are a central part of the field known as **operations research**. In their classic textbook Introduction to Operations Research, Hillier and Lieberman tell us:

> The development of linear programming has been ranked among the most important scientific advances of the mid-20th century... Today it is a standard tool that has saved many thousands or millions of dollars for most companies or businesses of even moderate size in the various industrialized countries of the world... A major proportion of all scientific computation on computers is devoted to the use of linear programming. (1995; p. 25)

Hillier and Lieberman go on to describe how during the mid-1980's a linear programming team develop a "refinery linear programming system" for Citgo Petroleum Corporation to help determine the efficiency of their supply and distribution of petroleum. This system determined that at any given time Citgo had $116.5 million in excess inventory resulting in a $14 million per year loss in interest expenses alone!

The employment statistics in this area also give us some idea of the importance of these areas. In 1995 Hillier and Liberman reported:

> The U.S. Bureau of Labor Statistics predicted that OR [Operations Research] will be the third fastest- growing career area for U.S. college graduates from 1990 to 2005. It is also predicted that 100,000 people will be employed as operations research analysts in the United States by the year 2005.

While there was no category for Operations Research Analysts, in the study by Careers.com that was reported in the *Wall Street Journal* on 26 Jaunary, 2009 (available online at:http://online.wsj.com/article/SB123119236117055127.html) the top three ranked jobs are:

1. Mathematician
2. Actuary
3. Statistician

The importance of mathematical tools like linear programming and job prospects for employment in related areas cannot be understated.

## The Lemonade and Cookies Problem

This problem is designed for elementary and middle school mathematics students. The problem is basic enough to be understood in its broad terms by this entire audience. While the mathematical analysis below will be too sophisticated in parts for some of these students, much of the data gathering, graphing, and computations can be done by students from K-8 with appropriate manupulatives, small group work, and teacher support.

These problems touch on many critical curriculum areas: arithmetic, problem solving, arithmetic sentences, fact families, beginning algebra, coordinate geometry, and algebraic geometry.

In fact, in some ways this is a model problem. Taught in the early grades it can be revisited throughout the curriculum, providing key connections in high school which includes equations of lines, solving systems of equations, graphing inequalities, and many other key topics in Algebra II.

Problem: You and a friend decide to want to open a lemonade stand for the weekend.

You think it would be good to sell chocolate chip cookies too.

Your Mom has generously donated the supplies to make the lemonade and cookies. There's enough lemon juice, chocolate chips, flour, eggs, butter, paper cups, and all the other supplies you need. The only problem is sugar – which you need a lot of. There are two ten pound bags available - about 42 cups of sugar.

Your Dad said he would help you make things on Friday after work. So there's a total of 5 hours to get ready.

You look in a cookbook and find out that a batch of lemonade takes 6 cups of sugar and makes about 40 12 oz. glasses of lemonade. A batch of cookies makes about 40 cookies and takes 2 cups of sugar.

You and your friend decide that 2 cookies should be the same price as a glass of lemonade, $0.25.

You and your friend want to make as much money as you can.

| |
|---|
| Question: Since sugar and time are both in short supply, how should you best allocate the sugar and time available between the lemonade and cookies? In other words, how many batches of lemonade should you make and how many batches of cookies should you make so you have the highest profit? |

The **constraints** of the problem are summarized in the table below:

| | **Lemonade** | **Cookies** | Total Available |
|---|---|---|---|
| Time per batch | 20 min. | 1 hour | 5 hours |
| Sugar per batch | 6 cups | 2 cups | 42 cups |
| Servings per batch | 40 | 20 | |
| Price per serving | $0.25 | $0.25 | |
| Profit per batch | $10.00 | $5.00 | |

Because you need to decide how many batches of lemonade and how many batches of cookies to make, these are called your **decision variables**.

Because your objective is to make the most profit, the profit is called your **objective function**.

Linear programming is a very geometric method for solving optimization problems. To illustrate it here we need to determine the relationships that the problem constraints impose on the decision variables.

Sugar Questions

1. If we make 3 batches of lemonade, how many batches of cookies can we make with the remaining sugar?
2. If we make 9 batches of cookies, how many batches of cookies can we make with the remaining sugar?
3. Can you find other combinations of sugar for cookies and lemonade?
4. What would we see if we graphed this data on a graph?

Answer: If we graph the points on a coordinate axes, we find that the points describe a line – as shown in Graph #1. As we increase the sugar spent making one thing the amount of time left for the other decreases linearly. This is why the method is called *linear* programming – all of the constraints must be linear.
We also see that the "Enough Sugar Zone" is triangular.

Time Questions

1. If we make 3 batches of cookies, how many batches of lemonade can we make with the remaining time?
2. If we make 3 batches of lemonade, how many batches of cookies can we make with the remaining time?
3. Can you find other combinations of time for cookies and time for lemonade?
4. What would we see if we graphed this data on a graph?

Answer: If we graph the points on a coordinate axes, we find that the points describe a line – as shown in Graph #2. As we increase the time spent making one thing the amount of time left for the other decreases linearly, and conversely. This is why the method is called *linear* programming – all of the constraints must be linear.
We also see that the "Enough Time Zone" is triangular.

If we graph the sugar constraint and the time constraint on the same axes we get three zones as pictured on Graph #3. Only one of these zones gives potential solutions to our problem as there is not enough sugar for points in the upper, left triangular section and there is not enough time in the lower, right triangular section. The section containing possible solutions is the quadrilateral at the origin. This region is called the **feasible region**.

Now that we know which solutions are feasible, we need to determine how to make the most profit. We do this much as above, by graphing.

Profit Questions

1. Can you find some combination of the decision variables which yield a profit of $10. If so, graph these.
2. Can you find some combination of the decision variables which yield a profit of $20. If so, graph these.
3. Can you find some combination of the decision variables which yield a profit of $30. If so, graph these.
4. Can you find some combination of the decision variables which yield a profit of $40. If so, graph these.

Answer: In each case you can find at least two combinations. If you graph these data you get the family of lines shown in Graph #4 which is a blown-up version of Graph #3 with the new linear data added.

They key observation now is that all of the profit lines are parallel. As the profit increases, the profit line is moved parallel in a north-northeasterly direction. How far can we move it? Well, as seen in Graph #5, the last profit line that remains in contact with the feasible region is P = $75. In other words, if we try to make more profit than $75 we will run out of sugar or time or both. So $75 is the maximum profit.

And how many batches should we make to make $75 in profit? Well, we simply read it off the graph. At that point the decision variables give the coordinate (3,6), so we make 3 batches of cookies and 6 batches of lemonade.

## Linear Programming and Obaminoes

The material here was developed to help provide a substantive connection to the Obaminoes mural that we made. (See obaminoes.wsc.ma.edu for more information.)

While this example is quite simple, the more general method works in the same way. In general, problems solved by linear programming many have 1,000 decision variables and

1,000,000 constraint equations.  The feasible region is then a polyhedron in 1,000 dimensions which likely has tens of thousands of faces.  But geometrically we can determine which is its thousands (or even millions) of vertices provides the optimal solution.

So how is the creation of Obaminoes a linear program? Well, our dominoes are the scarce resources - we have only a fixed number of each different tile configuration. Our objective is to find the optimal arrangement of dominoes so the difference between the grayscale of the domino arrangement and the pixilated image of Obama is minimized.